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ON A NEW METHOD FOR CALCULATING THE POTENTIAL FLOW PAST A BODY OF REVOLUTION

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SUMMARY

A new method is presented for obtaining the velocity potential of the flow about a solid of revolution moving uniformly in the direction of its axis of symmetry in a fluid otherwise at rest. This method is based essentially on the fact that the form of the differential equation for the velocity potential is invariant with regard to conformal transformations in a meridian plane. By means of the conformal transformation of the meridian profile into a circle a system of orthogonal curvilinear coordinates is obtained, the main feature of which is that one of the coordinate lines is the meridian profile itself. The use of this type of coordinate system yields a simple expression of the boundary condition at the surface of the solid and leads to a rational process of iteration for the solution of the differential equation for the velocity potential. It is shown that the velocity potential for an arbitrary body of revolution may be expressed in terms of universal functions which, although not normal, are obtainable by means of simple quadratures.

The general results are applied to a body of revolution obtained by revolving a symmetrical Joukowski profile about its axis of symmetry. A numerical example further serves to illustrate the theory.

INTRODUCTION

The simplest case of a three-dimensional fluid motion occurs when a body of revolution moves with a constant velocity in the direction of its axis of symmetry. In this case, the motion is the same in any plane passing through the axis of symmetry and, in this respect, presents some analogy with a two-dimensional motion. Thus, a stream function is defined by means of the equation of continuity; and the condition for irrotational motion yields a velocity potential. The stream function and the velocity potential, however, are not interchangeable in three-dimensional flows in the same way as are the corresponding quantities in two-dimensional irrotational motions. The reason for this difference is that, although the differential equation for the velocity potential is Laplace's equation, the equation for the stream function is not Laplace's and therefore the two functions cannot be combined to give an analytic function of a single complex variable. It follows that the elegant and powerful methods of the complex variable are not obviously applicable and the calculation of the potential flow past a body of revolution has, of necessity, developed along other lines.

The method of calculating the flow past a body of revolution most often referred to was suggested by Rankine and developed by von Kármán (reference 1) and others. For axial flow, the axis of the body is covered by a continuous distribution of sources and sinks in such a way that the closed stream surface found by the superposition of the flow induced by the sources and sinks on the parallel flow coincides with the surface of the solid. For transverse flow, the axis of the body is covered by a continuous distribution of doublets. The superposition of the flow due to the doublets on the parallel flow then yields the surface of the solid as a stream surface. Both problems lead to integral equations that von Kármán solved by an approximate method. As von Kármán pointed out, however, the exact replacement of the body by a distribution of singularities along the axis of symmetry is possible only when the analytical continuation of the potential function, free from singularities in the space outside the body, can be extended to the axis of symmetry without encountering singular points. (See reference 1, p. 27.) Inasmuch as the question of when this analytical continuation is or is not possible has never been answered, numerical calculations may lead to incorrect results, particularly in the case of a body with a rather blunt nose.

In reference 2 an attempt was made to calculate the potential flow past a body of revolution according to the methods of potential theory. In that paper, Laplace's equation for the velocity potential is expressed in terms of elliptic-cylindrical coordinates and is solved in conjunction with the appropriate boundary conditions for axial and transverse flows. In the case of an ellipsoid or a hyperboloid of revolution the solution is obtained in a closed form, both shapes being members of the family of orthogonal coordinate surfaces belonging to a system of elliptic-cylindrical coordinates. For any other solid of revolution, however, the method leads to two sets of linear equations, each set having an infinite number of equations and an infinite number of unknown coefficients for the determination of the velocity potentials for the axial and transverse flows.

In the present paper, a new method is presented for calculating the potential flow past an arbitrary body of revolution. Only the case of axial flow is discussed but the method is equally applicable to the case of transverse flow. The method is based on the discovery that, by the proper choice of the system of coordinates to be used for a given body of revolution, the solution of the potential-flow problem can be obtained by means of quadratures. Coordinate systems

of this nature already exist in the literature. For example, if the solid is an ellipsoid or a hyperboloid of revolution, the coordinate system used is an elliptic-cylindrical one. These coordinate systems possess in common the property that one of their coordinate surfaces is the boundary of the solid itself. It will be shown in this paper that, by means of the conformal transformation of the meridian profile into a circle, a system of orthogonal curvilinear coordinates can be defined such that one of the coordinate lines is the meridian profile of the solid of revolution. Furthermore, it will be shown that, by the use of this type of coordinate system, the potential-flow problem for an arbitrary body of revolution can be solved by means of elementary integrations.

In comparison of the method of this paper with the method of reference 2 it is to be recalled that in reference 2 Laplace's equation for the velocity potential was expressed in elliptic-cylindrical coordinates. The general solution involves normal functions: namely, Legendre functions of the first and second kinds. The boundary condition, however, for an arbitrary body of revolution involves the two independent variables of the problem and yields, as mentioned before, an infinite set of linear equations with an infinite number of unknown coefficients. On the other hand, the method of this paper utilizes a different coordinate system for each body of revolution. The main feature of these *conformal* orthogonal curvilinear coordinates is that one of the coordinate lines coincides with the meridian profile of the body of revolution. This fact leads to an expression of the boundary condition at the surface involving only one independent variable. Although the general solution of the differential equation for the velocity potential has not been obtained in terms of normal functions, a method of iteration has been devised that involves only elementary integrations. This solution, satisfying the boundary conditions, can then be expressed in terms of universal functions. Although these functions are not normal, they need be determined but once.

MATHEMATICAL DEVELOPMENTS

EQUATION FOR THE VELOCITY POTENTIAL

The axis of symmetry of the body is denoted by x and the position of a point in a meridian plane is fixed by the Cartesian coordinates (x, ϖ) (fig. 1). Then if q_x and q_ϖ are,

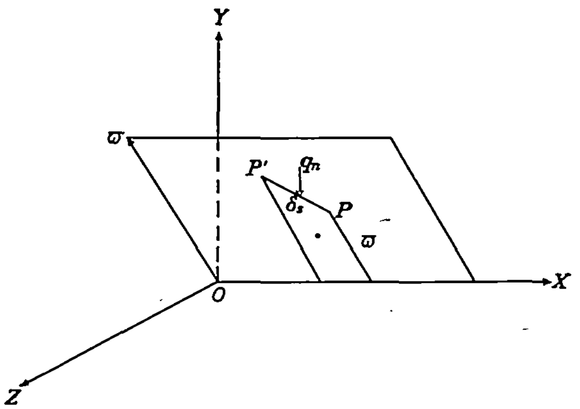

FIGURE 1.—The meridian plane $X0\varpi$

respectively, the components of the fluid velocity in the directions of the x and ϖ axes, the equation of continuity is obtained by equating to zero the flow out of the annular space obtained by revolving a small rectangle $dx\, d\varpi$ around the axis of symmetry. Thus, the total flow outward in the direction of the x axis is $\frac{\partial}{\partial x}(2\pi\varpi q_x d\varpi)dx$ and in the direction of the ϖ axis is $\frac{\partial}{\partial \varpi}(2\pi\varpi q_\varpi dx)d\varpi$. The equation of continuity is therefore

$$\frac{\partial}{\partial x}(\varpi q_x)+\frac{\partial}{\partial \varpi}(\varpi q_\varpi)=0 \tag{1}$$

Since the flow is symmetrical about the x axis, the vorticity is

$$\frac{\partial q_\varpi}{\partial x}-\frac{\partial q_x}{\partial \varpi}$$

and if, further, the motion is irrotational, then

$$\frac{\partial q_\varpi}{\partial x}-\frac{\partial q_x}{\partial \varpi}=0$$

A velocity potential ϕ can therefore be so defined that

$$\left.\begin{aligned} q_x&=-\frac{\partial \phi}{\partial x}\\ q_\varpi&=-\frac{\partial \phi}{\partial \varpi}\end{aligned}\right\} \tag{2}$$

and the equation of continuity (1) becomes

$$\frac{\partial}{\partial x}\left(\varpi\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial \varpi}\left(\varpi\frac{\partial\phi}{\partial \varpi}\right)=0 \tag{3}$$

Consider now the conjugate complex variables $z=x+i\varpi$ and $\bar{z}=x-i\varpi$. Then, symbolically,

$$2\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-i\frac{\partial}{\partial \varpi}$$

$$2\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}+i\frac{\partial}{\partial \varpi}$$

and it can be easily verified that

$$\frac{\partial}{\partial x}\left(\varpi\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial \varpi}\left(\varpi\frac{\partial\phi}{\partial \varpi}\right)=R.P.\,4\frac{\partial}{\partial z}\left(\varpi\frac{\partial\phi}{\partial \bar{z}}\right)$$

Thus, the vanishing of the real part of $\frac{\partial}{\partial z}\left(\varpi\frac{\partial\phi}{\partial \bar{z}}\right)$ is equivalent to equation (3).

Consider further the conformal transformation

$$z=f(\zeta) \text{ where } \zeta=\xi+i\eta$$

Then

$$\frac{\partial}{\partial x}\left(\varpi\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial \varpi}\left(\varpi\frac{\partial\phi}{\partial \varpi}\right)=R.P.\,4\frac{d\zeta}{dz}\frac{\partial}{\partial \zeta}\left(\varpi\frac{d\bar{\zeta}}{d\bar{z}}\frac{\partial\phi}{\partial \bar{\zeta}}\right) \tag{4}$$

Since z, $\bar{z}$, or ζ, $\bar{\zeta}$ may be looked upon as independent variables, it follows that the right-hand side of equation (4) is

$$R.P.\,4\frac{d\zeta}{dz}\frac{d\bar{\zeta}}{d\bar{z}}\frac{\partial}{\partial \zeta}\left(\varpi\frac{\partial\phi}{\partial \bar{\zeta}}\right)$$

Now, the product $\frac{d\zeta}{dz}\frac{d\bar{\zeta}}{d\bar{z}}$ is a real quantity. The vanishing of the real part of $\frac{\partial}{\partial\zeta}\left(\varpi\frac{\partial\phi}{\partial\bar{\zeta}}\right)$ is therefore equivalent to equation (3). Hence,

$$\frac{\partial}{\partial\xi}\left(\varpi\frac{\partial\phi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\varpi\frac{\partial\phi}{\partial\eta}\right)=0 \tag{5}$$

where ϖ is now a function of (ξ, η) obtained from the conformal transformation $z=f(\zeta)$.[1]

Equation (5) rather than equation (3) will be considered to be the fundamental differential equation governing axisymmetrical motions of a perfect incompressible fluid. It is more general than equation (3) in the sense that the independent variables (ξ, η) denote any set of orthogonal curvilinear coordinates obtained by means of a conformal transformation $z=f(\zeta)$. In addition, its form is invariant with regard to conformal transformations and, in this sense, does not complicate the original equation (3). In the following section it will be shown that, by means of the conformal transformation of the meridian profile of an arbitrary body of revolution into a circle, an orthogonal curvilinear system of coordinates (ξ, η) can be so defined that the coordinate line $\eta=0$ is the meridian profile itself. In subsequent sections it will be seen that the use of this type of coordinate system leads to a simple expression of the boundary condition and, consequently, to an iteration process for the solution of equation (5) for an arbitrary body of revolution.

CONFORMAL TRANSFORMATION AND ORTHOGONAL COORDINATES

It is well known that a unique conformal transformation exists which maps the region external to a given boundary in the z plane into the region external to a circle in the Z plane with its center at the origin, such that the regions at infinity of the two planes correspond. The function representing this conformal transformation can be developed in the region external to the circle in a convergent series of the type

$$z=Z+c_1+\frac{a_1}{Z}+\frac{a_2}{Z^2}+\frac{a_3}{Z^3}+\ \ldots \tag{6}$$

where the coefficients $c_1, a_1, a_2, a_3, \ldots$ are, in general, complex quantities. In this paper, the meridian profile is symmetrical with respect to the axis of revolution and these coefficients, therefore, are all *real*. The constants a_1, a_2, $a_3, \ldots$ depend only on the shape of the meridian profile in the z plane.

The method of Theodorsen and Garrick, described in reference 3, is particularly well adapted for the purpose of determining transformation (6). It is shown in reference 3 that, with the proper choice of axes and origin, the Joukowski transformation

$$z=Z'+\frac{a^2}{Z'} \tag{7}$$

maps a closed boundary in the z plane into a nearly circular curve in the Z' plane. The mapping of this nearly circular boundary, with its center at the origin of the Z plane, is then completed by means of the transformation

$$Z'=Ze^{f(Z)} \tag{8}$$

where

$$f(Z)=\sum_{n=1}^{\infty}\frac{c_n}{Z^n}$$

On elimination of Z' between equations (7) and (8), equation (6) follows, where

$$a_1=c_2+\frac{1}{2}c_1^2+a^2$$

$$a_2=c_3+c_2c_1+\frac{1}{6}c_1^3-c_1a^2$$

$$a_3=c_4+c_1c_3+\frac{1}{2}c_1^2c_2+\frac{1}{2}c_2^2+\frac{1}{24}c_1^4+\frac{1}{2}a^2c_1^2-a^2c_2$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

The values for c_n and therefore for a_n can actually be determined as the Fourier coefficients of a certain $\psi(\phi)$ curve (reference 3).

If the radius of the circle in the Z plane is denoted by R, the coordinates of any point on this circle can be expressed as

$$X=R\cos\xi$$

$$Y=-R\sin\xi$$

where, as ξ increases from 0 to 2π, the point describes the circle in a clockwise sense, thus leaving the external region to the left of the direction in which the boundary is traversed. Any point of the circle can also be expressed as

$$Z=X+iY=Re^{-i\xi}$$

If $\zeta=\xi+i\eta$, the transformation

$$Z=Re^{-i\zeta}$$

yields the circle of radius R for $\eta=0$ and, furthermore, $\eta=\infty$ corresponds to the region at infinity of the Z plane. Upon substitution of this expression for Z in equation (6), it follows that

$$z=Re^{\eta}e^{-i\xi}+c_1+\frac{a_1}{R}e^{-\eta}e^{i\xi}+\frac{a_2}{R^2}e^{-2\eta}e^{2i\xi}+\frac{a_3}{R^3}e^{-3\eta}e^{3i\xi}+\ \ldots \tag{9}$$

[1] It has been pointed out by Dr. Theodore Theodorsen that equation (5) can be obtained directly, as follows:

The vector form of equation (3) in a meridian plane is simply

$$\mathrm{div}\ (\varpi\ \overline{\mathrm{grad}}\ \phi)=0$$

If a set of orthogonal curvilinear coordinates ξ and η is introduced, the expression for div ($\varpi\ \overline{\mathrm{grad}}\ \phi$) becomes

$$\frac{1}{h_1h_2}\left[\frac{\partial}{\partial\xi}\left(\varpi\frac{h_2}{h_1}\frac{\partial\phi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\varpi\frac{h_1}{h_2}\frac{\partial\phi}{\partial\eta}\right)\right]$$

where the elements of length along the ξ-variable and η-variable coordinate lines are, respectively, $h_1\ d\xi$ and $h_2\ d\eta$.

If, however, the transformation from the rectangular Cartesian coordinates (x,ϖ) to the orthogonal curvilinear coordinates (ξ,η) is also conformal, then $h_1=h_2$; because for a conformal transformation the magnification at any point in the plane is independent of the direction through the point. Thus, with $h_1=h_2$, the foregoing expression placed equal to zero yields equation (5).

When this equation is separated into its real and imaginary parts, it is seen that

$$\left.\begin{aligned} x&=c_1+Re^{\eta}\cos\xi+\frac{a_1}{R}e^{-\eta}\cos\xi+\frac{a_2}{R^2}e^{-2\eta}\cos 2\xi \\ &\qquad+\frac{a_3}{R^3}e^{-3\eta}\cos 3\xi+\ldots \\ \varpi&=-Re^{\eta}\sin\xi+\frac{a_1}{R}e^{-\eta}\sin\xi+\frac{a_2}{R^2}e^{-2\eta}\sin 2\xi \\ &\qquad+\frac{a_3}{R^3}e^{-3\eta}\sin 3\xi+\ldots \end{aligned}\right\} \tag{10}$$

where the quantities c_1, a_1, a_2, a_3, . . . are determined according to the method of reference 3.

Rather than as a conformal transformation of a plane z into a plane Z, equations (10) are to be looked upon as the equations of transformation from the rectangular Cartesian coordinates (x, ϖ) into the orthogonal curvilinear coordinates (ξ, η). Furthermore, the coordinate line $\eta=0$ is the profile itself; that is, when $\eta=0$, equations (10) yield the parametric equations of the profile in the plane z. It will be seen in the following discussion that the use of this type of coordinate system leads to a simple form for the boundary condition at the solid surface.

BOUNDARY CONDITIONS

If a body of revolution moves with a velocity U in the direction of its axis of symmetry, the normal velocities of the body and the fluid in contact with it are the same. Thus, at the surface of the moving body, the boundary condition expressed in the *conformal* orthogonal coordinates (ξ,η) is simply

$$-\left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}=U\left(\frac{\partial x}{\partial\eta}\right)_{\eta=0}$$

since the coordinate lines along which η varies and ξ remains constant are normal to the boundary $\eta=0$. According to the first of equations (10) this boundary condition can be written as

$$-\left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}=U\left(R\cos\xi-\frac{a_1}{R}\cos\xi -\frac{2a_2}{R^2}\cos 2\xi-\frac{3a_3}{R^3}\cos 3\xi-\ldots\right) \tag{11}$$

Furthermore, as the fluid at infinity, originally at rest, remains undisturbed by the motion of the body, the boundary condition there is, simply,

$$\left.\begin{aligned} \left(\frac{\partial\phi}{\partial\xi}\right)_{\eta=\infty}&=0 \\ \left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=\infty}&=0 \end{aligned}\right\} \tag{12}$$

It is remarked that the simple form of the boundary condition (11) has been attained on account of the introduction of *conformal* orthogonal coordinates (ξ, η) so defined that one of the coordinate lines $\eta=0$ is the meridian profile itself. It will be seen in the following discussion that this choice of coordinate system leads to a process of iteration for the solution of the fundamental differential equation (5) involving only simple quadratures.

SOLUTION OF THE FUNDAMENTAL DIFFERENTIAL EQUATION BY ITERATION

If the right-hand side of equation (10) for ϖ is substituted into equation (5), it follows that

$$\left(Re^{\eta}-\frac{a_1}{R}e^{-\eta}\right)\sin\xi\left(\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}\right) +\left(Re^{\eta}-\frac{a_1}{R}e^{-\eta}\right)\cos\xi\frac{\partial\phi}{\partial\xi}+\left(Re^{\eta}+\frac{a_1}{R}e^{-\eta}\right)\sin\xi\frac{\partial\phi}{\partial\eta}= \sum_{n=2}^{\infty}\frac{a_n}{R^n}e^{-n\eta}\left[\sin n\xi\left(\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}\right)+n\cos n\xi\frac{\partial\phi}{\partial\xi}-n\sin n\xi\frac{\partial\phi}{\partial\eta}\right] \tag{13}$$

With regard to an iteration method it is desirable that the initial step in the process be obtained in a closed form. Thus, equation (13) has been so arranged that the solution of the left-hand side placed equal to zero can be obtained in a closed form. This initial solution can then be utilized as the starting point of an iteration process. Before a detailed description of the iteration method is given, however, it is first necessary to introduce several new parameters. Thus, the coefficient a_1 is replaced by $a^2e^{2\beta}$ and the radius R by ae^{α} where a, as in equation (7), serves merely to preserve dimensions. Then equation (13) can be written as follows:

$$\sinh(\eta+\alpha-\beta)\sin\xi\left(\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}\right)+\sinh(\eta+\alpha-\beta)\cos\xi\frac{\partial\phi}{\partial\xi} +\cosh(\eta+\alpha-\beta)\sin\xi\frac{\partial\phi}{\partial\eta}=\sum_{n=1}^{\infty}b_n e^{-(n+1)(\eta+\alpha-\beta)} \times\left[\sin(n+1)\xi\left(\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}\right)+(n+1)\cos(n+1)\xi\frac{\partial\phi}{\partial\xi} -(n+1)\sin(n+1)\xi\frac{\partial\phi}{\partial\eta}\right] \tag{14}$$

where $b_n=\frac{a_{n+1}}{2(ae^{\beta})^{n+2}}$, a nondimensional quantity, and the boundary condition (11) becomes

$$-\left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}=2aUe^{\beta}\left[\sinh(\alpha-\beta)\cos\xi -2b_1e^{-2(\alpha-\beta)}\cos 2\xi-3b_2e^{-3(\alpha-\beta)}\cos 3\xi-\ldots\right] \tag{15}$$

The method of solving equation (14) is based upon the following considerations:

It is assumed that the velocity potential ϕ can be developed in a series of terms each of which is homogeneous with respect to the indices of the coefficients b_n; that is,

$$\phi=\sum_{n=0}^{\infty}\phi_n \tag{16}$$

where, for example,

ϕ_0 involves *none* of the coefficients b_n

ϕ_1 contains the coefficient b_1 as a factor

ϕ_2 is the sum of two terms having, respectively, b_1^2 and b_2 as factors

ϕ_3 is the sum of three terms having, respectively, b_1^3, b_1b_2, and b_3 as factors and so on for the higher-order terms

Equation (16) for ϕ is now substituted into equation (14) and the terms involving the coefficients b_n and their products to the same degree in the indices are equated. The same process is applied to the boundary condition (15). In this manner, equation (14) is replaced by a set of partial differential equations which, in conjunction with the boundary conditions, can be solved rigorously for $\phi_0, \phi_1, \phi_2, \ldots$ When the operations just described are performed, the differential equations with the accompanying boundary conditions for the first three functions ϕ_0, ϕ_1, and ϕ_2 are as follows:

$$\sinh(\eta+\alpha-\beta)\sin\xi\left(\frac{\partial^2\phi_0}{\partial\xi^2}+\frac{\partial^2\phi_0}{\partial\eta^2}\right)+\sinh(\eta+\alpha-\beta)\cos\xi\frac{\partial\phi_0}{\partial\xi}$$
$$+\cosh(\eta+\alpha-\beta)\sin\xi\frac{\partial\phi_0}{\partial\eta}=0 \tag{17}$$

with the boundary condition at the surface

$$-\left(\frac{\partial\phi_0}{\partial\eta}\right)_{\eta=0}=2aUe^{\beta}\sinh(\alpha-\beta)\cos\xi$$

and at infinity

$$\left(\frac{\partial\phi_0}{\partial\xi}\right)_{\eta=\infty}=0 \quad\text{and}\quad \left(\frac{\partial\phi_0}{\partial\eta}\right)_{\eta=\infty}=0$$

$$\sinh(\eta+\alpha-\beta)\sin\xi\left(\frac{\partial^2\phi_1}{\partial\xi^2}+\frac{\partial^2\phi_1}{\partial\eta^2}\right)$$
$$+\sinh(\eta+\alpha-\beta)\cos\xi\frac{\partial\phi_1}{\partial\xi}+\cosh(\eta+\alpha-\beta)\sin\xi\frac{\partial\phi_1}{\partial\eta}=$$
$$b_1e^{-2(\eta+\alpha-\beta)}\left[\sin 2\xi\left(\frac{\partial^2\phi_0}{\partial\xi^2}+\frac{\partial^2\phi_0}{\partial\eta^2}\right)\right.$$
$$\left.+2\cos 2\xi\frac{\partial\phi_0}{\partial\xi}-2\sin 2\xi\frac{\partial\phi_0}{\partial\eta}\right] \tag{18}$$

with the boundary condition at the surface

$$\left(\frac{\partial\phi_1}{\partial\eta}\right)_{\eta=0}=4aUe^{\beta}b_1e^{-2(\alpha-\beta)}\cos 2\xi$$

and at infinity

$$\left(\frac{\partial\phi_1}{\partial\xi}\right)_{\eta=\infty}=0 \quad\text{and}\quad \left(\frac{\partial\phi_1}{\partial\eta}\right)_{\eta=\infty}=0$$

$$\sinh(\eta+\alpha-\beta)\sin\xi\left(\frac{\partial^2\phi_2}{\partial\xi^2}+\frac{\partial^2\phi_2}{\partial\eta^2}\right)$$
$$+\sinh(\eta+\alpha-\beta)\cos\xi\frac{\partial\phi_2}{\partial\xi}+\cosh(\eta+\alpha-\beta)\sin\xi\frac{\partial\phi_2}{\partial\eta}=$$
$$b_1e^{-2(\eta+\alpha-\beta)}\left[\sin 2\xi\left(\frac{\partial^2\phi_1}{\partial\xi^2}+\frac{\partial^2\phi_1}{\partial\eta^2}\right)+2\cos 2\xi\frac{\partial\phi_1}{\partial\xi}-2\sin 2\xi\frac{\partial\phi_1}{\partial\eta}\right]$$
$$+b_2e^{-3(\eta+\alpha-\beta)}\left[\sin 3\xi\left(\frac{\partial^2\phi_0}{\partial\xi^2}+\frac{\partial^2\phi_0}{\partial\eta^2}\right)+3\cos 3\xi\frac{\partial\phi_0}{\partial\xi}-3\sin 3\xi\frac{\partial\phi_0}{\partial\eta}\right] \tag{19}$$

with the boundary condition at the surface

$$\left(\frac{\partial\phi_2}{\partial\eta}\right)_{\eta=0}=6aUe^{\beta}b_2e^{-3(\alpha-\beta)}\cos 3\xi$$

and at infinity

$$\left(\frac{\partial\phi_2}{\partial\xi}\right)_{\eta=\infty}=0 \quad\text{and}\quad \left(\frac{\partial\phi_2}{\partial\eta}\right)_{\eta=\infty}=0$$

DETERMINATION OF ϕ_0

In order to solve equation (17) for ϕ_0, it is convenient to introduce a new set of independent variables μ and λ, where

$$\mu=\cos\xi \quad\text{and}\quad \lambda=\cosh(\eta+\alpha-\beta)$$

By use of the symbolic expressions

$$\left.\begin{aligned}
&\frac{\partial}{\partial\xi}=-\sin\xi\frac{\partial}{\partial\mu}\\
&\frac{\partial^2}{\partial\xi^2}=-\cos\xi\frac{\partial}{\partial\mu}+\sin^2\xi\frac{\partial^2}{\partial\mu^2}\\
&\text{and}\\
&\frac{\partial}{\partial\eta}=\sinh(\eta+\alpha-\beta)\frac{\partial}{\partial\lambda}\\
&\frac{\partial^2}{\partial\eta^2}=\cosh(\eta+\alpha-\beta)\frac{\partial}{\partial\lambda}+\sinh^2(\eta+\alpha-\beta)\frac{\partial^2}{\partial\lambda^2}
\end{aligned}\right\} \tag{20}$$

it follows that equation (17) can be written as

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi_0}{\partial\mu}\right]-\frac{\partial}{\partial\lambda}\left[(1-\lambda^2)\frac{\partial\phi_0}{\partial\lambda}\right]=0 \tag{21}$$

If a solution of the form $\phi_0=F(\mu)\,G(\lambda)$ is assumed, equation (21) becomes

$$\frac{1}{F}\frac{d}{d\mu}\left[(1-\mu^2)\frac{dF}{d\mu}\right]=\frac{1}{G}\frac{d}{d\lambda}\left[(1-\lambda^2)\frac{dG}{d\lambda}\right]$$

Since the left-hand side of this equation contains μ but not λ and the right-hand side contains λ but not μ, $F(\mu)$ and $G(\lambda)$ must be such that each side is a constant. If, furthermore, the constant is chosen to be $-n(n+1)$, where n is a positive integer, then

$$\left.\begin{aligned}
&\frac{d}{d\mu}\left[(1-\mu^2)\frac{dF}{d\mu}\right]+n(n+1)F=0\\
&\frac{d}{d\lambda}\left[(1-\lambda^2)\frac{dG}{d\lambda}\right]+n(n+1)G=0
\end{aligned}\right\} \tag{22}$$

Each of these equations is a Legendre differential equation and, being of the second order, possesses two independent particular solutions: namely, the Legendre functions of the first and second kinds, P_n and Q_n, respectively. According to the boundary condition at the surface, the form to be taken for $F(\mu)$ is $P_1(\mu)$. Correspondingly, according to the boundary condition at infinity, the form to be taken for $G(\lambda)$ is $Q_1(\lambda)$, since the $Q_n(\lambda)$ and their derivatives vanish for $\lambda=\infty$. It follows that the solution of equation (21) is

$$\phi_0=A_1P_1(\mu)Q_1(\lambda) \tag{23}$$

The arbitrary constant A_1 is determined by the boundary condition at the surface: namely,

$$-\left(\frac{\partial\phi_0}{\partial\lambda}\right)_{\lambda=\lambda_0}=2aUe^{\beta}\mu$$

Thus,

$$A_1=-\frac{2aUe^{\beta}}{Q_1'(\lambda_0)}$$

where

$$Q_1'(\lambda_0)=\left[\frac{dQ_1(\lambda)}{d\lambda}\right]_{\lambda=\lambda_0}$$

and

$$\lambda_0=\cosh(\alpha-\beta)$$

It will be necessary for the determination of ϕ_1 and ϕ_2 to have the first few orders of $P_n(\mu)$ and $Q_n(\lambda)$ stated explicitly. The following are expressions for the first four harmonics:

$$\left.\begin{aligned}
P_0(\mu)&=1\\
P_1(\mu)&=\mu=\cos\xi\\
P_2(\mu)&=\frac{1}{2}(3\mu^2-1)=\frac{3}{4}\cos 2\xi+\frac{1}{4}\\
P_3(\mu)&=\frac{1}{2}(5\mu^3-3\mu)=\frac{5}{8}\cos 3\xi+\frac{3}{8}\cos\xi\\
Q_0(\lambda)&=\frac{1}{2}\log\frac{\lambda+1}{\lambda-1}\\
Q_1(\lambda)&=\frac{1}{2}\lambda\log\frac{\lambda+1}{\lambda-1}-1\\
Q_2(\lambda)&=\frac{1}{4}(3\lambda^2-1)\log\frac{\lambda+1}{\lambda-1}-\frac{3}{2}\lambda\\
Q_3(\lambda)&=\frac{1}{4}(5\lambda^3-3\lambda)\log\frac{\lambda+1}{\lambda-1}-\frac{5}{2}\lambda^2+\frac{2}{3}
\end{aligned}\right\}\qquad(24)$$

DETERMINATION OF ϕ_1

When the independent variables μ and λ are introduced and equations (21) and (23) are used, it follows that equation (18) for ϕ_1 can be written as

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi_1}{\partial\mu}\right]-\frac{\partial}{\partial\lambda}\left[(1-\lambda^2)\frac{\partial\phi_1}{\partial\lambda}\right]=$$
$$\frac{2}{3}b_1A_1\frac{(\lambda-\sqrt{\lambda^2-1})^2}{\sqrt{\lambda^2-1}}\{[2Q_1(\lambda)-(\lambda+2\sqrt{\lambda^2-1})Q_1'(\lambda)]P_0(\mu)$$
$$-2[Q_1(\lambda)+(\lambda+2\sqrt{\lambda^2-1})Q_1'(\lambda)]P_2(\mu)\}\qquad(25)$$

The right-hand side of this equation requires that

$$\phi_1=F_0(\lambda)P_0(\mu)+F_2(\lambda)P_2(\mu)\qquad(26)$$

If this expression for ϕ_1 is substituted into equation (25) and the coefficients of $P_0(\mu)$ and $P_2(\mu)$ are equated on both sides of the resulting equation, it follows that

$$\frac{d}{d\lambda}\left[(1-\lambda^2)\frac{dF_0(\lambda)}{d\lambda}\right]=$$
$$\frac{2}{3}b_1A_1\frac{(\lambda-\sqrt{\lambda^2-1})^2}{\sqrt{\lambda^2-1}}[(\lambda+2\sqrt{\lambda^2-1})\,Q_1'(\lambda)-2Q_1(\lambda)]\qquad(27)$$

and

$$\frac{d}{d\lambda}\left[(1-\lambda^2)\frac{dF_2(\lambda)}{d\lambda}\right]+6F_2(\lambda)=$$
$$\frac{4}{3}b_1A_1\frac{(\lambda-\sqrt{\lambda^2-1})^2}{\sqrt{\lambda^2-1}}[(\lambda+2\sqrt{\lambda^2-1})\,Q_1'(\lambda)+Q_1(\lambda)]\qquad(28)$$

With the right-hand sides taken equal to zero, these two equations reduce, respectively, to Legendre's equation for the zero- and second-order functions. Then, according to the theory of linear differential equations of the second order, the solutions of equations (27) and (28) are:

$$F_0(\lambda)=B_0Q_0(\lambda)-\int\frac{d\lambda}{\lambda^2-1}\int R_0(\lambda)d\lambda\qquad(29)$$

and

$$F_2(\lambda)=B_2Q_2(\lambda)-(3\lambda^2-1)\int\frac{d\lambda}{(\lambda^2-1)(3\lambda^2-1)^2}\int(3\lambda^2-1)R_2(\lambda)d\lambda\qquad(30)$$

where $R_0(\lambda)$ and $R_2(\lambda)$ are, respectively, the right-hand sides of equations (27) and (28). It is noted that each of these solutions contains only one arbitrary constant, B_0 and B_2, because the other two independent solutions $P_0(\lambda)$ and $P_2(\lambda)$, respectively, of the homogeneous forms of equations (27) and (28) do not satisfy the boundary condition at infinity and therefore the arbitrary constants associated with them are taken equal to zero.

The boundary conditions for ϕ_1 lead to the following boundary conditions for $F_0(\lambda)$ and $F_2(\lambda)$:

$$\left.\begin{aligned}
\left(\frac{dF_0}{d\lambda}\right)_{\lambda=\lambda_0}&=-\frac{4}{3}b_1aUe^{\beta}\frac{(\lambda_0-\sqrt{\lambda_0^2-1})^2}{\sqrt{\lambda_0^2-1}}\\
(F_0)_{\lambda=\infty}&=\left(\frac{dF_0}{d\lambda}\right)_{\lambda=\infty}=0
\end{aligned}\right\}\qquad(31)$$

and

$$\left.\begin{aligned}
\left(\frac{dF_2}{d\lambda}\right)_{\lambda=\lambda_0}&=\frac{16}{3}b_1aUe^{\beta}\frac{(\lambda_0-\sqrt{\lambda_0^2-1})^2}{\sqrt{\lambda_0^2-1}}\\
(F_2)_{\lambda=\infty}&=\left(\frac{dF_2}{d\lambda}\right)_{\lambda=\infty}=0
\end{aligned}\right\}\qquad(32)$$

The integrations required in equations (29) and (30) are straightforward and need not be performed here. Furthermore, by use of the well-known recurrence formulas for the Legendre functions, it can be shown that

$$F_0(\lambda)=-\frac{2}{3}b_1A_1\left\{\frac{2}{3}\left[2Q_0(\lambda)+Q_2(\lambda)\right]+\sqrt{\lambda^2-1}\left[Q_0'(\lambda)-Q_1(\lambda)\right]\right\}\qquad(33)$$

and

$$F_2(\lambda)=B_2Q_2(\lambda)-\frac{4}{3}b_1A_1\left\{\frac{1}{6}\left[2Q_0(\lambda)+Q_2(\lambda)\right]+\sqrt{\lambda^2-1}\left[Q_0'(\lambda)+2Q_1(\lambda)\right]\right\}\qquad(34)$$

where, by means of the boundary conditions, equations (31) and (32),

$$\left.\begin{aligned}
B_0&=0\\
B_2&=6b_1A_1\frac{\lambda_0}{Q_2'(\lambda_0)}\left\{Q_1'(\lambda_0)+\frac{2}{3}\sqrt{\lambda_0^2-1}[Q_0'(\lambda_0)]^2\right\}
\end{aligned}\right\}\qquad(35)$$

It may be easily verified that these expressions for $F_0(\lambda)$ and $F_2(\lambda)$ satisfy the condition that they and their derivatives vanish for $\lambda=\infty$.

DETERMINATION OF ϕ_2

When the variables μ and λ are introduced into equations (21) and (25), it follows that equation (19) for ϕ_2 can be written as

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi_2}{\partial\mu}\right]-\frac{\partial}{\partial\lambda}\left[(1-\lambda^2)\frac{\partial\phi_2}{\partial\lambda}\right]=2b_1\frac{(\lambda-\sqrt{\lambda^2-1})^2}{\sqrt{\lambda^2-1}}\left[(1-\mu^2)\frac{\partial\phi_1}{\partial\mu}-\mu(\lambda+2\sqrt{\lambda^2-1})\frac{\partial\phi_1}{\partial\lambda}\right]$$
$$+b_2\frac{(\lambda-\sqrt{\lambda^2-1})^3}{\sqrt{\lambda^2-1}}\left[8\mu(1-\mu^2)\frac{\partial\phi_0}{\partial\mu}-(4\mu^2-1)(\lambda+3\sqrt{\lambda^2-1})\frac{\partial\phi_0}{\partial\lambda}\right]+4b_1^2\frac{(\lambda-\sqrt{\lambda^2-1})^4}{\lambda^2-1}\left[\mu(1-\mu^2)\frac{\partial\phi_0}{\partial\mu}-\mu^2(\lambda+2\sqrt{\lambda^2-1})\frac{\partial\phi_0}{\partial\lambda}\right] \tag{36}$$

If the expressions for ϕ_0 and ϕ_1 given, respectively, by equations (23) and (26) are substituted in equation (36), it follows that:

$$\frac{\partial}{\partial\lambda}\left[(1-\lambda^2)\frac{\partial\phi_2}{\partial\lambda}\right]-\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi_2}{\partial\mu}\right]=R_1(\lambda)P_1(\mu)+R_3(\lambda)P_3(\mu) \tag{37}$$

where

$$R_1(\lambda)=-2b_1\frac{(\lambda-\sqrt{\lambda^2-1})^2}{\sqrt{\lambda^2-1}}\left\{\frac{6}{5}F_2(\lambda)-(\lambda+2\sqrt{\lambda^2-1})\left[F_0'(\lambda)+\frac{2}{5}F_2'(\lambda)\right]\right\}$$
$$-\frac{2}{5}b_2A_1\frac{(\lambda-\sqrt{\lambda^2-1})^3}{\sqrt{\lambda^2-1}}\left[8Q_1(\lambda)-\frac{7}{2}(\lambda+3\sqrt{\lambda^2-1})Q_1'(\lambda)\right]-\frac{4}{5}b_1^2A_1\frac{(\lambda-\sqrt{\lambda^2-1})^4}{\lambda^2-1}\left[2Q_1(\lambda)-3(\lambda+2\sqrt{\lambda^2-1})Q_1'(\lambda)\right].$$

and

$$R_3(\lambda)=\frac{6}{5}b_1\frac{(\lambda-\sqrt{\lambda^2-1})^2}{\sqrt{\lambda^2-1}}\left[2F_2(\lambda)+(\lambda+2\sqrt{\lambda^2-1})F_2'(\lambda)\right]+\frac{8}{5}b_2A_1\frac{(\lambda-\sqrt{\lambda^2-1})^3}{\sqrt{\lambda^2-1}}\left[2Q_1(\lambda)+(\lambda+3\sqrt{\lambda^2-1})Q_1'(\lambda)\right]$$
$$+\frac{8}{5}b_1^2A_1\frac{(\lambda-\sqrt{\lambda^2-1})^4}{\lambda^2-1}\left[Q_1(\lambda)+(\lambda+2\sqrt{\lambda^2-1})Q_1'(\lambda)\right]$$

The right-hand side of equation (37) requires that

$$\phi_2=F_1(\lambda)P_1(\mu)+F_3(\lambda)P_3(\mu) \tag{38}$$

When this expression for ϕ_2 is substituted into equation (37) and the coefficients of $P_1(\mu)$ and $P_3(\lambda)$ are equated on both sides of the resulting equation, the following equations for $F_1(\lambda)$ and $F_3(\lambda)$ are obtained:

$$\frac{d}{d\lambda}\left[(1-\lambda^2)\frac{dF_1(\lambda)}{d\lambda}\right]+2F_1(\lambda)=R_1(\lambda) \tag{39}$$

and

$$\frac{d}{d\lambda}\left[(1-\lambda^2)\frac{dF_3(\lambda)}{d\lambda}\right]+12F_3(\lambda)=R_3(\lambda) \tag{40}$$

It is evident that the homogeneous forms of these equations are, respectively, Legendre's equations for the first- and third-order functions. According, then, to the theory of linear differential equations of the second order, the solutions of equations (39) and (40) are

$$F_1(\lambda)=B_1Q_1(\lambda)-\int\frac{d\lambda}{\lambda^2(\lambda^2-1)}\int R_1(\lambda)d\lambda \tag{41}$$

and

$$F_3(\lambda)=B_3Q_3(\lambda)$$
$$-(5\lambda^3-3\lambda)\int\frac{d\lambda}{(\lambda^2-1)(5\lambda^3-3\lambda)^2}\int(5\lambda^3-3\lambda)R_3(\lambda)d\lambda \tag{42}$$

The arbitrary constants B_1 and B_3 are determined by the boundary condition at the surface of the moving body. Thus, the boundary condition for ϕ_2 leads to the following conditions for $F_1(\lambda)$ and $F_3(\lambda)$:

$$\left.\begin{aligned}\left(\frac{dF_1}{d\lambda}\right)_{\lambda=\lambda_0}&=-\frac{18}{5}b_2aUe^{\beta}\frac{(\lambda_0-\sqrt{\lambda_0^2-1})^3}{\sqrt{\lambda_0^2-1}}\\(F_1)_{\lambda=\infty}&=\left(\frac{dF_1}{d\lambda}\right)_{\lambda=\infty}=0\end{aligned}\right\} \tag{43}$$

and

$$\left.\begin{aligned}\left(\frac{dF_3}{d\lambda}\right)_{\lambda=\lambda_0}&=\frac{48}{5}b_2aUe^{\beta}\frac{(\lambda_0-\sqrt{\lambda_0^2-1})^3}{\sqrt{\lambda_0^2-1}}\\(F_3)_{\lambda=\infty}&=\left(\frac{dF_3}{d\lambda}\right)_{\lambda=\infty}=0\end{aligned}\right\} \tag{44}$$

Then, if the integrations required in equations (41) and (42) are made and the recurrence formulas for the Legendre functions are used, it can be shown that

$$F_1(\lambda)=B_1Q_1(\lambda)-\frac{4}{5}b_1B_2\left\{\frac{2}{5}Q_1(\lambda)+\frac{3}{5}Q_3(\lambda)+\sqrt{\lambda^2-1}[Q_1'(\lambda)-Q_2(\lambda)]\right\}$$
$$-\frac{8}{5}b_1^2A_1\left\{-4-\frac{43}{20}Q_1(\lambda)-\frac{3}{5}Q_3(\lambda)+\frac{3}{4}Q_0'(\lambda)+\sqrt{\lambda^2-1}[Q_2(\lambda)+4Q_0(\lambda)-5Q_1'(\lambda)]\right\}$$
$$-\frac{8}{5}b_2A_1\left\{1+\frac{7}{5}Q_1(\lambda)+\frac{3}{5}Q_3(\lambda)-\sqrt{\lambda^2-1}\left[Q_2(\lambda)+Q_0(\lambda)-\frac{7}{8}Q_1'(\lambda)\right]\right\} \tag{45}$$

and

$$F_3(\lambda)=B_3Q_3(\lambda)-\frac{6}{5}b_1B_2\left\{\frac{2}{5}Q_1(\lambda)+\frac{3}{5}Q_3(\lambda)+\sqrt{\lambda^2-1}[Q_1'(\lambda)+4Q_2(\lambda)]\right\}$$
$$-\frac{8}{5}b_1^2A_1\left\{-1-\frac{3}{5}Q_1(\lambda)+\frac{17}{20}Q_3(\lambda)+\frac{1}{2}Q_0'(\lambda)+\sqrt{\lambda^2-1}\left[-6Q_2(\lambda)+Q_0(\lambda)-\frac{5}{2}Q_1'(\lambda)\right]\right\}$$
$$-\frac{8}{5}b_2A_1\left\{\frac{2}{3}+\frac{3}{5}Q_1(\lambda)-\frac{1}{10}Q_3(\lambda)+\sqrt{\lambda^2-1}\left[\frac{8}{3}Q_2(\lambda)-\frac{2}{3}Q_0(\lambda)+Q_1'(\lambda)\right]\right\} \tag{46}$$

where

$$B_1Q_1'(\lambda_0)=\frac{4}{5}b_1B_2\left(3Q_2(\lambda_0)+Q_1'(\lambda_0)-\frac{3}{2}\sqrt{\lambda_0^2-1}\left\{3Q_1(\lambda_0)+Q_0'(\lambda_0)-\frac{2}{3}[Q_0'(\lambda_0)]^2\right\}\right)$$
$$-\frac{8}{5}b_1^2A_1\left(3Q_2(\lambda_0)+\frac{11}{4}Q_1'(\lambda_0)-\frac{3}{4}Q_0''(\lambda_0)-\frac{3}{2}\sqrt{\lambda_0^2-1}\left\{3Q_1(\lambda_0)+Q_0'(\lambda_0)-\frac{10}{3}[Q_0'(\lambda_0)]^2\right\}\right)$$
$$-\frac{8}{5}b_2A_1\left(-3Q_0(\lambda)+\frac{11}{8}Q_1'(\lambda_0)-\sqrt{\lambda_0^2-1}\left\{3Q_0'(\lambda_0)-\frac{1}{4}[Q_0'(\lambda_0)]^2\right\}\right) \tag{47}$$

and

$$B_3Q_3'(\lambda_0)=\frac{6}{5}b_1B_2\left\{3Q_2(\lambda_0)+Q_1'(\lambda_0)-\frac{1}{\sqrt{\lambda_0^2-1}}\left[\frac{11}{5}Q_1(\lambda_0)-\frac{36}{5}Q_3(\lambda_0)+Q_0'(\lambda_0)\right]\right\}$$
$$+\frac{8}{5}b_1^2A_1\left\{\frac{17}{4}Q_2(\lambda_0)+\frac{1}{4}Q_1'(\lambda_0)+\frac{1}{2}Q_0''(\lambda_0)-\frac{1}{\sqrt{\lambda_0^2-1}}\left[-\frac{33}{10}Q_1(\lambda_0)+\frac{54}{5}Q_3(\lambda_0)-\frac{5}{2}Q_0'(\lambda_0)\right]\right\}$$
$$+\frac{8}{5}b_2A_1\left\{\frac{15}{2}Q_2(\lambda_0)+\frac{19}{2}Q_1'(\lambda_0)-8Q_0(\lambda_0)-\frac{4}{\sqrt{\lambda_0^2-1}}[Q_0'(\lambda_0)-2]\right\} \tag{48}$$

It may be easily verified that the expressions for $F_1(\lambda)$ and $F_3(\lambda)$ and their derivatives vanish for $\lambda=\infty$.

It is clear that the iteration can be continued to obtain further members of the sequence of ϕ_n. The order n to which the process must be carried depends on the magnitude of the coefficients b_n of the conformal transformation. For example, it may be that one of the higher coefficients, say b_m, is significant. It is obvious, then, that the iteration must be continued far enough to include at least the corresponding term ϕ_m. Although the ϕ_n's do not form a set of normal functions (obtained independently of one another), they are nevertheless uniquely determined by means of simple quadratures; that is, they constitute a set of universal functions in the sense that the labor necessary for their determination is performed but once and need not be repeated for each particular body of revolution.

VELOCITY AND PRESSURE

According to Bernoulli's theorem,

$$\frac{p-p_\infty}{\frac{1}{2}\rho U^2}=1-\left(\frac{q_r}{U}\right)^2 \tag{49}$$

where

- p pressure anywhere in fluid
- p_∞ pressure in region at infinity undisturbed by motion of solid
- ρ density of fluid
- q_r velocity of fluid relative to boundary of solid

It is recalled that in this paper the expression for the velocity potential ϕ has been derived by considering the coordinate axes as rigidly attached to the body and therefore moving with a velocity U in the positive direction of the x axis. The velocity vector $\bar{q}=-\overline{\text{grad}}\ \phi$ represents the velocity of the fluid relative to the undisturbed fluid at infinity. The vector velocity $\bar{q}_r$, relative to the moving axes, is therefore given by

$$\bar{q}_r=\bar{q}-\overline{U}$$

or

$$\bar{q}_r=\bar{i}(q_x-U)+\bar{j}q_\varpi$$

where $\bar{i}$ and $\bar{j}$ are, respectively, unit vectors along the positive directions of the x and ϖ axes. The magnitude of the relative velocity is therefore obtained from

$$q_r^2=q_x^2+q_\varpi^2-2Uq_x+U^2 \tag{50}$$

where

$$q_x=-\frac{\partial\phi}{\partial x} \text{ and } q_\varpi=-\frac{\partial\phi}{\partial\varpi}$$

It is recalled, however, that the velocity potential ϕ is derived as a function of the independent variables ξ and η. In order then to determine q_r, it is necessary to express the velocity components q_x and q_ϖ in terms of the quantities $\frac{\partial\phi}{\partial\xi}$ and $\frac{\partial\phi}{\partial\eta}$. These expressions can be derived in the following way:

The relations

$$z=x+i\varpi \quad \text{and} \quad \bar{z}=x-i\varpi$$

can be considered as equations of transformation from the coordinates (x, ϖ) to the independent coordinates $(z, \bar{z})$. It follows then, symbolically, that

$$2\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-i\frac{\partial}{\partial \varpi}$$

and

$$2\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}+i\frac{\partial}{\partial \varpi}$$

Therefore

$$q_x+iq_\varpi=-\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial \varpi}\right)\phi=-2\frac{\partial\phi}{\partial \bar{z}}$$

where ϕ is now a function of the independent variables z and $\bar{z}$.

The conformal transformation $z=f(\zeta)$ then gives

$$q_x+iq_\varpi=-2\frac{\partial\phi}{\partial\bar{\zeta}}\frac{d\bar{\zeta}}{d\bar{z}}$$

or

$$q_x+iq_\varpi=-\frac{2}{J^2}\frac{dz}{d\zeta}\frac{\partial\phi}{d\bar{\zeta}} \tag{51}$$

where

$$J^2=\frac{dz}{d\zeta}\frac{d\bar{z}}{d\bar{\zeta}}$$

Symbolically,

$$2\frac{\partial}{\partial\zeta}=\frac{\partial}{\partial\xi}-i\frac{\partial}{\partial\eta}$$

and therefore

$$2\frac{dz}{d\zeta}=\left(\frac{\partial x}{\partial\xi}+\frac{\partial\varpi}{\partial\eta}\right)+i\left(\frac{\partial\varpi}{\partial\xi}-\frac{\partial x}{\partial\eta}\right)$$

Then, by means of the Cauchy-Riemann relations

$$\frac{\partial x}{\partial\xi}=\frac{\partial\varpi}{\partial\eta} \quad \text{and} \quad \frac{\partial x}{\partial\eta}=-\frac{\partial\varpi}{\partial\xi}$$

it follows that

$$\frac{dz}{d\zeta}=\frac{\partial x}{\partial\xi}-i\frac{\partial x}{\partial\eta}$$

or

$$\frac{dz}{d\zeta}=\frac{\partial\varpi}{\partial\eta}+i\frac{\partial\varpi}{\partial\xi}$$

Equation (51) therefore becomes

$$\begin{aligned}q_x+iq_\varpi&=-\frac{1}{J^2}\left(\frac{\partial x}{\partial\xi}-i\frac{\partial x}{\partial\eta}\right)\left(\frac{\partial\phi}{\partial\xi}+i\frac{\partial\phi}{\partial\eta}\right)\\&=-\frac{1}{J^2}\left(\frac{\partial\varpi}{\partial\eta}+i\frac{\partial\varpi}{\partial\xi}\right)\left(\frac{\partial\phi}{\partial\xi}+i\frac{\partial\phi}{\partial\eta}\right)\end{aligned} \tag{52}$$

where

$$J^2=\left(\frac{\partial x}{\partial\xi}\right)^2+\left(\frac{\partial x}{\partial\eta}\right)^2=\left(\frac{\partial\varpi}{\partial\xi}\right)^2+\left(\frac{\partial\varpi}{\partial\eta}\right)^2$$

When the real and imaginary parts on both sides of equation (52) are equated, it follows that

$$\left.\begin{aligned}q_x&=-\frac{1}{J^2}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial x}{\partial\xi}+\frac{\partial\phi}{\partial\eta}\frac{\partial x}{\partial\eta}\right)\\q_\varpi&=-\frac{1}{J^2}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\varpi}{\partial\xi}+\frac{\partial\phi}{\partial\eta}\frac{\partial\varpi}{\partial\eta}\right)\end{aligned}\right\} \tag{53}$$

By means of these equations q_r^2 and therefore the pressure coefficient $\frac{p-p_\infty}{\frac{1}{2}\rho U^2}$ given, respectively, by equations (50) and (49) can be obtained as functions of the independent variables ξ and η.

APPLICATION OF THEORETICAL RESULTS

SYMMETRICAL JOUKOWSKI SHAPE

General expressions.—The theoretical results of this paper will now be applied to the case of a body of revolution whose meridian section is a symmetrical Joukowski profile with both rounded nose and tail. This example is sufficiently complicated to illustrate the principles and usefulness of the method.

It is well known that by means of the mapping function

$$z=Z'+\frac{a^2}{Z'} \tag{54}$$

the circle of radius a, with its center at the origin of the Z' plane, is transformed into the line segment extending from $z=-2a$ to $z=2a$ in the z plane; and the circle of radius $a(1+\epsilon_1+\epsilon_2)$ with its center at $Z'=\epsilon_1 a$ is transformed into a symmetrical Joukowski profile with rounded nose and tail in the z plane (fig. 2). When Z' is replaced by $Z+\epsilon_1 a$, equation (54) becomes

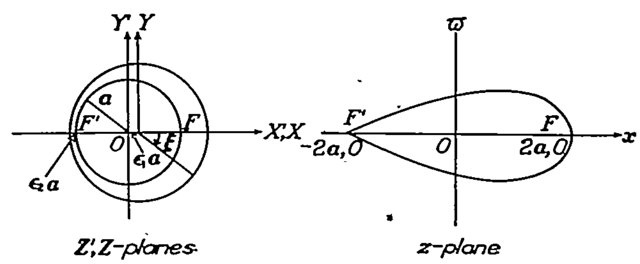

FIGURE 2.—Mapping of a circle into a symmetrical Joukowski profile.

$$z=Z+\epsilon_1 a+\frac{a^2}{Z+\epsilon_1 a}$$

or

$$z=Z+\epsilon_1 a+\frac{a^2}{Z}-\epsilon_1\frac{a^3}{Z^2}+\epsilon_1^2\frac{a^4}{Z^3}-\epsilon_1^3\frac{a^5}{Z^4}+\ .\ .\ . \tag{55}$$

A comparison of this equation with equation (6) shows that

$$c_1=\epsilon_1 a,\ a_1=a^2,\ a_2=-\epsilon_1 a^3,\ a_3=\epsilon_1^2 a^4,\ a_4=-\epsilon_1^3 a^5,\ .\ .\ .$$

Also, the radius of the circle in the Z plane being denoted by ae^α, it follows that

$$e^\alpha=1+\epsilon_1+\epsilon_2 \tag{56}$$

According to definition

$$e^{2\beta}=\frac{a_1}{a^2}\text{ and }b_m=\frac{a_{m+1}}{2(ae^\beta)^{m+2}}$$

where

$$m \geqq 1$$

As $a_1=a^2$ and $a_{m+1}=(-1)^m\epsilon_1^m a^{m+2}$, it follows therefore that

$$\beta=0\text{ and }b_m=\frac{1}{2}(-1)^m\epsilon_1^m \tag{57}$$

The expression for the velocity potential then becomes

$$\phi=\phi_0+\phi_1+\phi_2+\ \ldots \tag{58}$$

where, from equation (23)

$$\phi_0=A_1P_1(\mu)Q_1(\lambda) \tag{59}$$

with

$$A_1=-\frac{2aU}{Q_1'(\lambda_0)}$$

and

$$\lambda_0=\cosh\alpha=\frac{1}{2}\left(1+\epsilon_1+\epsilon_2+\frac{1}{1+\epsilon_1+\epsilon_2}\right)$$

From equation (26)

$$\phi_1=F_0(\lambda)P_0(\mu)+F_2(\lambda)P_2(\mu) \tag{60}$$

where

$$F_0(\lambda)=\frac{1}{3}\epsilon_1A_1\left\{\frac{2}{3}[2Q_0(\lambda)+Q_2(\lambda)]+\sqrt{\lambda^2-1}[Q_0'(\lambda)-Q_1(\lambda)]\right\}$$

$$F_2(\lambda)=B_2Q_2(\lambda)+\frac{2}{3}\epsilon_1A_1\left\{\frac{1}{6}[2Q_0(\lambda)+Q_2(\lambda)]+\sqrt{\lambda^2-1}[Q_0'(\lambda)+2Q_1(\lambda)]\right\}$$

and

$$B_2=-\epsilon_1A_1\frac{\lambda_0}{Q_2'(\lambda_0)}\left\{3Q_1'(\lambda_0)+2\sqrt{\lambda_0^2-1}[Q_0'(\lambda_0)]^2\right\}$$

From equation (38),

$$\phi_2=F_1(\lambda)P_1(\mu)+F_3(\lambda)P_3(\mu) \tag{61}$$

where

$$F_1(\lambda)=B_1Q_1(\lambda)+\frac{2}{5}\epsilon_1B_2\left\{\frac{1}{5}[2Q_1(\lambda)+3Q_3(\lambda)]+\sqrt{\lambda^2-1}\ [Q_1'(\lambda)-Q_3(\lambda)]\right\}$$
$$+\frac{2}{5}\epsilon_1^2A_1\left\{2+\frac{8}{5}Q_1(\lambda)-\frac{3}{5}Q_3(\lambda)-\frac{3}{4}Q_2'(\lambda)+\sqrt{\lambda^2-1}\left[-2Q_0(\lambda)+Q_2(\lambda)+\frac{13}{4}Q_1'(\lambda)\right]\right\}$$

$$F_3(\lambda)=B_3Q_3(\lambda)+\frac{3}{5}\epsilon_1B_2\left\{\frac{1}{5}[2Q_1(\lambda)+3Q_3(\lambda)]+\sqrt{\lambda^2-1}\ [Q_1'(\lambda)+4Q_2(\lambda)]\right\}$$
$$+\frac{2}{5}\epsilon_1^2A_1\left\{-\frac{1}{3}-\frac{3}{5}Q_1(\lambda)-\frac{13}{20}Q_3(\lambda)-\frac{1}{2}Q_0'(\lambda)+\sqrt{\lambda^2-1}\left[\frac{1}{3}Q_0(\lambda)+\frac{2}{3}Q_2(\lambda)+\frac{1}{2}Q_1'(\lambda)\right]\right\}$$

and

$$B_1Q_1'(\lambda_0)=-\frac{2}{5}\epsilon_1B_2\left(3Q_2(\lambda_0)+Q_1'(\lambda_0)-\frac{3}{2}\sqrt{\lambda_0^2-1}\{3Q_1(\lambda_0)+Q_0'(\lambda_0)-\frac{2}{3}[Q_0'(\lambda_0)]^2\}\right)$$
$$-\frac{2}{5}\epsilon_1^2A_1\left(-6Q_0(\lambda_0)+3Q_2(\lambda_0)+\frac{11}{2}Q_1'(\lambda_0)-\frac{3}{4}Q_0''(\lambda_0)-\frac{1}{2}\sqrt{\lambda_0^2-1}\{9Q_1(\lambda_0)+15Q_0'(\lambda_0)-11[Q_0'(\lambda_0)]^2\}\right)$$

$$B_3Q_3'(\lambda_0)=-\frac{3}{5}\epsilon_1B_2\left\{3Q_2(\lambda_0)+Q_1'(\lambda_0)-\frac{1}{\sqrt{\lambda_0^2-1}}\left[\frac{11}{5}Q_1(\lambda_0)-\frac{36}{5}Q_3(\lambda_0)+Q_0'(\lambda_0)\right]\right\}$$
$$-\frac{2}{5}\epsilon_1A_1\left\{16Q_0(\lambda_0)-\frac{77}{4}Q_2(\lambda_0)-\frac{77}{4}Q_1'(\lambda_0)-\frac{1}{2}Q_0''(\lambda_0)-\frac{1}{\sqrt{\lambda_0^2-1}}\left[16+\frac{33}{10}Q_1(\lambda_0)-\frac{54}{5}Q_3(\lambda_0)-\frac{11}{2}Q_0'(\lambda_0)\right]\right\}$$

Velocity and pressure.—From equation (54) together with the transformations

$$Z'=\epsilon_1a+Z\text{ and }Z=ae^{\alpha}e^{-i\zeta}$$

it follows that

$$\left.\begin{aligned}x&=a(\epsilon_1+e^{\eta+\alpha}\cos\xi)\left(1+\frac{1}{\epsilon_1^2+2\epsilon_1e^{\eta+\alpha}\cos\xi+e^{2(\eta+\alpha)}}\right)\\ \varpi&=-ae^{\eta+\alpha}\sin\xi\left(1-\frac{1}{\epsilon_1^2+2\epsilon_1e^{\eta+\alpha}\cos\xi+e^{2(\eta+\alpha)}}\right)\end{aligned}\right\} \tag{62}$$

Then

$$\frac{\partial x}{\partial\xi}=-ae^{\eta+\alpha}\sin\xi\left\{1-\frac{\epsilon_1^2-e^{2(\eta+\alpha)}}{[\epsilon_1^2+2\epsilon_1e^{\eta+\alpha}\cos\xi+e^{2(\eta+\alpha)}]^2}\right\}$$

and

$$\frac{\partial x}{\partial\eta}=ae^{\eta+\alpha}\cos\xi-\frac{[\epsilon_1^2+e^{2(\eta+\alpha)}]e^{\eta+\alpha}\cos\xi+2\epsilon_1e^{2(\eta+\alpha)}}{[\epsilon_1^2+2\epsilon_1e^{\eta+\alpha}\cos\xi+e^{2(\eta+\alpha)}]^2}$$

At the surface of the body $\eta=0$, so that

$$\left.\begin{aligned}\left(\frac{\partial x}{\partial\xi}\right)_{\eta=0}&=-ae^{\alpha}\sin\xi\left[1-\frac{\epsilon_1^2-e^{2\alpha}}{(\epsilon_1^2+2\epsilon_1e^{\alpha}\cos\xi+e^{2\alpha})^2}\right]\\ \left(\frac{\partial x}{\partial\eta}\right)_{\eta=0}&=ae^{\alpha}\cos\xi-\frac{(\epsilon_1^2+e^{2\alpha})e^{\alpha}\cos\xi+2\epsilon_1e^{2\alpha}}{(\epsilon_1^2+2\epsilon_1e^{\alpha}\cos\xi+e^{2\alpha})^2}\end{aligned}\right\} \tag{63}$$

Therefore

$$(J^2)_{\eta=0}=a^2e^{2\alpha}\left[1-\frac{2}{\epsilon_1^2+2\epsilon_1e^{\alpha}\cos\xi+e^{2\alpha}}+\frac{1+2e^{2\alpha}(1-\cos 2\xi)}{(\epsilon_1^2+2\epsilon_1e^{\alpha}\cos\xi+e^{2\alpha})^2}\right] \tag{64}$$

The boundary condition at the surface yields immediately the expression for $\left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}$. Thus

$$-\left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}=\left(U\frac{\partial x}{\partial\eta}\right)_{\eta=0} \tag{65}$$

where the expression for $\left(\frac{\partial x}{\partial\eta}\right)_{\eta=0}$ is given by the second of equations (63).

The expression for $\left(\frac{\partial\phi}{\partial\xi}\right)_{\eta=0}$ is obtained from the equation for ϕ. Thus, the velocity potential can be written as,

$$\phi=A_1P_1(\mu)Q_1(\lambda)+[F_0(\lambda)P_0(\mu)+F_2(\lambda)P_2(\mu)]+[F_1(\lambda)P_1(\mu)+F_3(\lambda)P_3(\mu)]+\ldots$$

Then

$$-\left(\frac{\partial\phi}{\partial\xi}\right)_{\eta=0}=\left[A_1Q_1(\lambda_0)+F_1(\lambda_0)+\frac{3}{8}F_3(\lambda_0)\right]\sin\xi+\frac{3}{2}F_2(\lambda_0)\sin 2\xi+\frac{15}{8}F_3(\lambda_0)\sin 3\xi-\ldots \tag{66}$$

where A_1, $F_1(\lambda_0)$, $F_2(\lambda_0)$, and $F_3(\lambda_0)$ are obtained from equations (59), (60), and (61).

By means of equations (63), (64), (65), and (66) the velocity q_r of the fluid and the pressure coefficient $\frac{p-p_\infty}{\frac{1}{2}\rho U^2}$ given, respectively, by equations (50) and (49) can be evaluated.

Numerical example.—As a numerical example, consider the case where $\epsilon_1=0.15$ and $\epsilon_2=0.10$. Then $e^\alpha=1+\epsilon_1+\epsilon_2=\frac{5}{4}$ and

$$\left.\begin{aligned}\left(\frac{x}{2a}\right)_{\eta=0}&=\frac{209}{120}+\frac{5}{8}\cos\xi-\frac{1540}{3}\frac{1}{317+75\cos\xi}\\ \left(\frac{\varpi}{2a}\right)_{\eta=0}&=\frac{5}{8}\sin\xi\left(-1+\frac{200}{317+75\cos\xi}\right)\end{aligned}\right\} \tag{67}$$

Also

$$\left.\begin{aligned}\frac{1}{2a}\left(\frac{\partial x}{\partial\xi}\right)_{\eta=0}&=-\frac{5}{8}\sin\xi\left[1+\frac{61600}{(317+75\cos\xi)^2}\right]\\ \frac{1}{2a}\left(\frac{\partial x}{\partial\eta}\right)_{\eta=0}&=\frac{5}{8}\left[\cos\xi-\frac{200(75+317\cos\xi)}{(317+75\cos\xi)^2}\right]\end{aligned}\right\} \tag{68}$$

and

$$\left(\frac{1}{2a}J\right)^2_{\eta=0}=\frac{25}{64}\left[1-\frac{400}{317+75\cos\xi}+\frac{5000(33-25\cos 2\xi)}{(317+75\cos\xi)^2}\right] \tag{69}$$

Now,

$$\lambda_0=\frac{e^\alpha+e^{-\alpha}}{2}=\frac{41}{40} \text{ and } \sqrt{\lambda_0^2-1}=\frac{e^\alpha-e^{-\alpha}}{2}=\frac{9}{40}$$

Therefore, from equations (24)

$$Q_0(\lambda_0)=2.19723;\ Q_0'(\lambda_0)=-\frac{1600}{81};\ Q_0''(\lambda_0)=\frac{5248}{6.561}$$

$$Q_1(\lambda_0)=1.25215;\ Q_1'(\lambda_0)=-18.04969$$

$$Q_2(\lambda_0)=0.82657;\ Q_2'(\lambda_0)=-15.99664$$

$$Q_3(\lambda_0)=0.57729;\ Q_3'(\lambda_0)=-13.91684$$

and from equations (59), (60), and (61)

$$A_1=0.055403(2aU)$$
$$B_1=0.502158\epsilon_1^2(2aU)$$
$$B_2=0.431087\epsilon_1(2aU)$$
$$B_3=0.537892\epsilon_1^2(2aU)$$
$$F_1(\lambda_0)=0.079143\epsilon_1^2(2aU)$$
$$F_2(\lambda_0)=0.245119\epsilon_1(2aU)$$
$$F_3(\lambda_0)=-0.180438\epsilon_1^2(2aU)$$

Then, from equation (66),

$$-\left(\frac{\partial\phi}{\partial\xi}\right)_{\eta=0}=2aU[(0.069373+0.011479\epsilon_1^2)\sin\xi+0.367679\epsilon_1\sin 2\xi-0.338321\epsilon_1^2\sin 3\xi]$$

or with $\epsilon_1=0.15$,

$$-\left(\frac{\partial\phi}{\partial\xi}\right)_{\eta=0}=2aU(0.069631\sin\xi+0.055152\sin 2\xi-0.007612\sin 3\xi) \tag{70}$$

From equations (65) and (68),

$$-\left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}=\frac{5}{8}(2aU)\left[\cos\xi-\frac{200(75+317\cos\xi)}{(317+75\cos\xi)^2}\right] \tag{71}$$

Table I presents the calculated values of the quantities given by equations (68), (69), (70), and (71) for intervals of 10° in the angle ξ. Table II gives the coordinates of the profile and the corresponding values of the velocity q_r/U and the pressure coefficient $\frac{p-p_\infty}{\frac{1}{2}\rho U^2}$. Figure 3 shows the

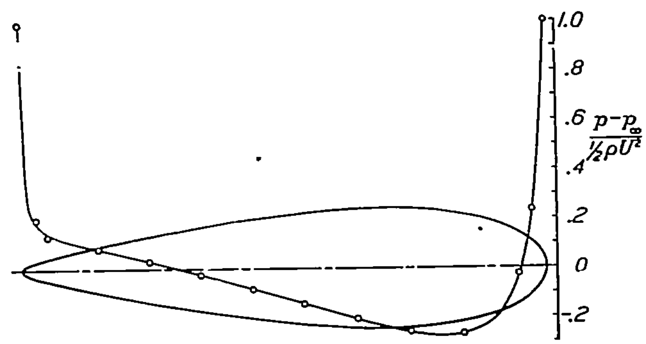

FIGURE 3.—Pressure distribution at the surface of a Joukowski shape.

graphs of the meridian profile and the pressure distribution. The small circles on this graph denote the values of $\frac{p-p_\infty}{\frac{1}{2}\rho U^2}$ obtained by the method of reference 2. It is seen that the numerical results obtained by the two methods agree almost perfectly.

LOW-DRAG SYMMETRICAL AIRFOIL SHAPE

The abscissas and ordinates of the meridian profile are given in table III. The first step is to obtain the $\psi(\phi)$ curve of the meridian profile according to the method of Theodorsen and Garrick (reference 3). The ϕ in reference 3 corresponds to the $-\xi$ of the present report. A Fourier analysis of the $\psi(\phi)$ curve is then performed and immediately yields the conformal transformation of the meridian profile into a circle. This conformal transformation is then looked upon as the equations of transformation from the rectangular Cartesian coordinates (x,ϖ) of the meridian plane into the orthogonal curvilinear coordinates (ξ,η), defined in such a manner that the coordinate line $\eta=0$ is the meridian profile itself. For $\eta=0$, then, these equations yield the parametric equations of the profile and for the present numerical example they are as follows:

$$\left.\begin{aligned}\frac{x}{2a}&=0.08895+0.98698\cos\xi-0.05586\cos 2\xi\\&\quad+0.03553\cos 3\xi-0.01122\cos 4\xi+\ldots\\\frac{\varpi}{2a}&=-0.25762\sin\xi-0.05586\sin 2\xi\\&\quad+0.03553\sin 3\xi-0.01122\sin 4\xi+\ldots\end{aligned}\right\}\tag{72}$$

The chord c of the profile is given by

$$c=2.0450a$$

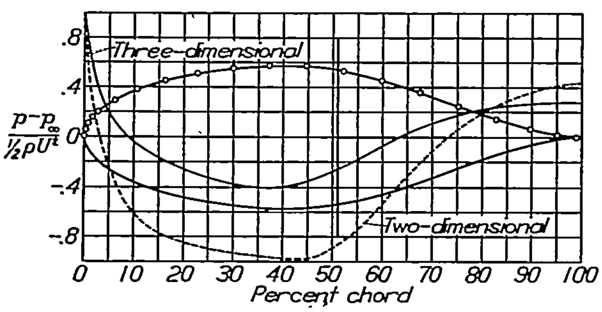

FIGURE 4.—Pressure-distribution curves.

TABLE I

CALCULATED VALUES FROM EQUATIONS (68,) (69), (70), AND (71)

ξ (deg)	$\frac{1}{2a}\left(\frac{\partial x}{\partial\xi}\right)_{\eta=0}$	$\frac{1}{2a}\left(\frac{\partial x}{\partial\eta}\right)_{\eta=0}$	$\frac{1}{2aU}\left(\frac{\partial\phi}{\partial\xi}\right)_{\eta=0}$	$\frac{1}{2aU}\left(\frac{\partial\phi}{\partial\eta}\right)_{\eta=0}$	$\left(\frac{1}{2a}J\right)^2_{\eta=0}$
0	0	0.30612	0	−0.30612	0.09371
5	−.07635	.30426	−.01368	−.30426	.09840
10	−.15229	.29871	−.02715	−.29871	.11242
15	−.22746	.28953	−.04022	−.28953	.13557
20	−.30145	.27686	−.05267	−.27686	.16753
30	−.44445	.24178	−.07497	−.24178	.25600
40	−.57824	.19543	−.09248	−.19543	.37255
50	−.69990	.14049	−.10385	−.14049	.50959
60	−.80658	.08025	−.10807	−.08025	.65702
70	−.89544	.01849	−.10469	−.01849	.80216
80	−.96362	−.04072	−.09403	.04072	.93022
90	−1.00813	−.09329	−.07724	.09329	1.02502
100	−1.02584	−.13552	−.05630	.13552	1.07071
110	−1.01351	−.16455	−.03379	.16455	1.05428
120	−.96807	−.17889	−.01254	.17889	.96917
130	−.88699	−.17896	.00478	.17896	.81876
140	−.76911	−.16735	.01615	.16735	.61953
150	−.61552	−.14866	.02056	.14866	.40096
160	−.43043	−.12888	.01823	.12888	.20188
170	−.22162	−.11399	.01058	.11399	.06211
175	−.11164	−.10988	.00548	.10988	.02454
180	0	−.10847	0	.10847	.01177

TABLE II

RELATIVE VELOCITY AND PRESSURE COEFFICIENTS CORRESPONDING TO PROFILE COORDINATES

ξ (deg)	$\frac{x}{2a}$	$\frac{\varpi}{2a}$	$\frac{q_r}{U}$	$\frac{p-p_\infty}{\frac{1}{2}\rho U^2}$
0	1.0572	0	0	1
5	1.0538	.0267	.28700	.91765
10	1.0438	.0530	.53519	.71357
15	1.0273	.0787	.72700	.47148
20	1.0042	.1034	.86521	.25141
30	.9390	.1489	1.02660	−.05390
40	.8496	.1872	1.09887	−.20751
50	.7378	.2166	1.12592	−.26709
60	.6061	.2359	1.12840	−.27330
70	.4573	.2445	1.11667	−.24606
80	.2948	.2425	1.09660	−.20253
90	.1223	.2307	1.07204	−.14927
100	−.0556	.2105	1.04580	−.09369
110	−.2340	.1842	1.02000	−.04036
120	−.4074	.1540	.99609	.00781
130	−.5699	.1225	.97500	.04933
140	−.7149	.0922	.95662	.08488
150	−.8363	.0645	.93905	.11718
160	−.9279	.0403	.91741	.15836
170	−.9851	.0193	.84684	.28287
175	−.9997	.0095	.67771	.54071
180	−1.0045	0	0	1

TABLE III

ABSCISSAS AND ORDINATES OF MERIDIAN PROFILE

x (percent chord)	ϖ (percent chord)
0	0
.50	2.08
.75	2.53
1.25	3.25
2.50	4.62
5.00	6.59
7.50	8.06
10.00	9.23
15.00	11.04
20.00	12.32
25.00	13.23
30.00	13.84
35.00	14.20
40.00	14.29
45.00	14.12
50.00	13.62
55.00	12.69
60.00	11.34
65.00	9.78
70.00	8.03
75.00	6.24
80.00	4.47
85.00	2.88
90.00	1.56
95.00	.54
100.00	0

Equations (72) may then be written as

$$\left.\begin{aligned}\frac{x}{c}&=0.04350+0.48263\cos\xi-0.02732\cos 2\xi\\&\quad+0.01737\cos 3\xi-0.00549\cos 4\xi+\ldots\\\frac{\varpi}{c}&=-0.12598\sin\xi-0.02732\sin 2\xi\\&\quad+0.01737\sin 3\xi-0.00549\sin 4\xi+\ldots\end{aligned}\right\}\tag{73}$$

Table IV gives values of $\frac{x}{c}$ and $\frac{\varpi}{c}$ calculated according to these equations and figure 4 shows the very good agreement between these points (small circles) and the profile plotted from table III (solid line).

With the coefficients of the conformal transformation known, the next step is to obtain an expression for the velocity potential ϕ at the surface of the body. For the present numerical case,

$$\begin{aligned}\phi=2aU(&0.00687+0.08181\cos\xi+0.03384\cos 2\xi\\&-0.02251\cos 3\xi+\ldots)\end{aligned}\tag{74}$$

TABLE IV

CALCULATED VALUES FROM EQUATIONS (72) AND (73)

ξ	$\sin \xi$	$\sin 2\xi$	$\sin 3\xi$	$\sin 4\xi$	$\cos \xi$	$\cos 2\xi$	$\cos 3\xi$	$\cos 4\xi$	$\frac{x}{c}$	$\frac{\varpi}{c}$
0	0	0	0	0	1	1	1	1	0.5107	0
5	.08716	.17365	.25882	.34202	.99619	.98481	.96593	.93969	.5090	−.0131
10	.17365	.34202	.50000	.64279	.98481	.93969	.86603	.76604	.5040	−.0261
15	.25882	.50000	.70711	.86603	.96593	.86603	.70711	.50000	.4956	−.0388
20	.34202	.64279	.86603	.98481	.93969	.76604	.50000	.17365	.4838	−.0510
30	.50000	.86603	1.0000	.86603	.86603	.50000	0	−.50000	.4506	−.0740
40	.64279	.98481	.86603	.34202	.76604	.17365	−.50000	−.93969	.4049	−.0947
50	.76604	.98481	.50000	−.34202	.64279	−.17365	−.86603	−.93969	.3486	−.1129
60	.86603	.86603	0	−.86603	.50000	−.50000	−1.0000	−.50000	.2838	−.1280
70	.93969	.64279	−.50000	−.98481	.34202	−.76604	−.86603	.17365	.2135	−.1392
80	.98481	.34202	−.86603	−.64279	.17365	−.93969	−.50000	.76604	.1401	−.1449
90	1.0000	0	−1.0000	0	0	−1.0000	0	1.0000	.0653	−.1434
100	.98481	−.34202	−.86603	.64279	−.17365	−.93969	.50000	.76604	−.0102	−.1333
110	.93969	−.64279	−.50000	.98481	−.34202	−.76604	.86603	.17365	−.0866	−.1149
120	.86603	−.86603	0	.86603	−.50000	−.50000	1	−.50000	−.1640	−.0902
130	.76604	−.98481	.50000	.34202	−.64279	−.17365	.86603	−.93969	−.2418	−.0628
140	.64279	−.98481	.86603	−.34202	−.76604	.17365	.50000	−.93969	−.3171	−.0372
150	.50000	−.86603	1.0000	−.86603	−.86603	.50000	0	−.50000	−.3854	−.0172
160	.34202	−.64279	.86603	−.98481	−.93969	.76604	−.50000	.17365	−.4406	−.0051
170	.17365	−.34202	.50000	−.64279	−.98481	.93969	−.86603	.76604	−.4767	−.0003
180	0	0	0	0	−1.0000	1.0000	−1.0000	1.0000	−.4893	0

TABLE V

CALCULATED VALUES FROM EQUATIONS (49) TO (53)

ξ	$\frac{\partial x}{\partial \xi}\Big/2a$	$\frac{\partial x}{\partial \eta}\Big/2a$	$\left(\frac{J}{2a}\right)^2$	$-\frac{\partial \phi}{\partial \xi}\Big/2aU$	$-\frac{\partial \phi}{\partial \eta}\Big/2aU$	$\frac{q_x}{U}$	$\frac{q_\varpi}{U}$	$\left(\frac{q_r}{U}\right)^2$	$\frac{q_r}{U}$	$\frac{p-p_\infty}{\tfrac{1}{2}\rho U^2}$
0	0	0.3077	0.0947	0	0.3077	1	0	0	0	1
15	−.2361	.2927	.1414	.0073	.2927	.5937	−.5037	.4188	.6472	.5812
20	−.3138	.2822	.1781	.0130	.2822	.4241	−.5178	.5998	.7745	.4002
30	−.4644	.2565	.2825	.0320	.2565	.1809	−.4524	.8755	.9357	.1245
40	−.6013	.2279	.4135	.0608	.2279	.0372	−.3648	1.060	1.030	−.0601
50	−.7147	.1963	.5493	.0956	.1963	−.0542	−.2896	1.195	1.093	−.1952
60	−.7969	.1571	.6597	.1295	.1571	−.1190	−.2206	1.301	1.141	−.3008
70	−.8466	.1026	.7272	.1541	.1026	−.1650	−.1412	1.377	1.174	−.3771
80	−.8703	.0274	.7582	.1622	.0274	−.1852	−.0374	1.406	1.186	−.4060
90	−.8804	−.0668	.7796	.1493	−.0668	−.1629	.0883	1.360	1.166	−.3602
100	−.8890	−.1686	.8188	.1159	−.1686	−.0911	.2069	1.233	1.111	−.2333
110	−.9018	−.2582	.8798	.0671	−.2582	.0070	.2843	1.067	1.033	−.0670
120	−.9126	−.3137	.9313	.0122	−.3137	.0937	.3115	.9185	.9584	.0815
130	−.9040	−.3195	.9193	−.0377	−.3195	.1481	.3011	.8163	.9034	.1837
140	−.8521	−.2734	.8008	−.0725	−.2734	.1705	.2662	.7589	.8709	.2412
150	−.7357	−.1897	.5772	−.0852	−.1897	.1710	.2188	.7330	.8561	.2670
160	−.5459	−.0954	.3071	−.0740	−.0954	.1612	.1466	.7251	.8515	.2749
170	−.2917	−.0220	.0856	−.0427	−.0220	.1512	.0641	.7246	.8512	.2754
180	0	.0056	.0000	0	.0056	1	0	0	0	1

The boundary condition at the surface of the moving body is simply

$$-\left(\frac{\partial \phi}{\partial \eta}\right)_{\eta=0}=U\left(\frac{\partial x}{\partial \eta}\right)_{\eta=0}$$

Since $\frac{\partial x}{\partial \eta}=-\frac{\partial \varpi}{\partial \xi}$, it follows from equations (72) that

$$-\left(\frac{\partial \phi}{\partial \eta}\right)_{\eta=0}=2aU(0.25762 \cos \xi+0.11172 \cos 2\xi -0.10658 \cos 3\xi+0.04489 \cos 4\xi- \ldots) \tag{75}$$

and from equation (74) that

$$-\left(\frac{\partial \phi}{\partial \xi}\right)_{\eta=0}=2aU(0.08181 \sin \xi+0.06768 \sin 2\xi -0.06752 \sin 3\xi+ \ldots) \tag{76}$$

Also, from equations (72),

$$\left.\begin{aligned} \frac{\partial x}{\partial \xi}=\frac{\partial \varpi}{\partial \eta}&=-0.98698 \sin \xi+0.11172 \sin 2\xi \\ &-0.10658 \sin 3\xi+0.04489 \sin 4\xi- \ldots \\ \frac{\partial x}{\partial \eta}=-\frac{\partial \varpi}{\partial \xi}&=0.25762 \cos \xi+0.11172 \cos 2\xi \\ &-0.10658 \cos 3\xi+0.04489 \cos 4\xi- \ldots \end{aligned}\right\} \tag{77}$$

Equations (75), (76), and (77) suffice to determine the pressure distribution over the surface of the body with the use of equations (49) to (53) contained in the section entitled "Velocity and Pressure." Table V presents the calculated values of the various quantities given by equations (49) to (53), the last column containing values of the pressure coefficient $\frac{p-p_\infty}{\frac{1}{2}\rho U^2}$. Figure 4 shows the graphs of the pressure distribution for both the body of revolution (solid curve) and the airfoil of infinite span having the meridian profile as cross section (broken curve).

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *May 18, 1942*.

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